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[redacted]

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MEMORANDUM

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Declass Review by NGA.

To: [redacted]
From:
Subject: On the Relationship Between Sine Wave Modulation and Some Coherent Optical Filtering Measurements

CC: [redacted]

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INTRODUCTION

In order to evaluate microdensitometers, it is desirable to have an alternate method for determining the modulation, M, of intensity transmittance recorded on film as

$$T = A\left(1 + M \sin \omega_0 x\right) \tag{1}$$

It has been proposed elsewhere[1] that the modulation can be determined by measurements on the spectrum of the corresponding amplitude transmittance, t, where

$$t = \sqrt{T} = \sqrt{A}\left(1 + M \sin \omega_0 x\right)^{1/2} \tag{2}$$

as obtained using coherent light. For small M, we have

$$t \approx \sqrt{A}\left(1 + \frac{M}{2} \sin \omega_0 x\right) \tag{3}$$

and it would appear that M can be determined from the ratio*

$$\lambda = \frac{\text{Intensity of Fundamental}}{\text{Intensity of DC}} = \frac{A\frac{M^2}{4}}{A} = \frac{M^2}{4} \tag{4}$$

* It should be noted that whereas the analysis herein uses single sided spectra, coherent optics displays the two sided spectrum. This means that only the intensity of one term of $\sin \omega_0 x = \frac{1}{2} e^{i\omega_0 x} + \frac{1}{2} e^{-i\omega_0 x}$ can be measured so that $M^2/4$ becomes $M^2/16$ as far as practical measurements are concerned.

It is of interest to determine the range of M for which the above approximate relationship is valid by considering a more exact expression such as might be obtained from the expansion of (2) in a Fourier series.

FOURIER SERIES

Without loss of generality, we can take $A=1$ and write

$$t = (1+M\sin\omega_0 x)^{1/2} = \frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m \cos m\omega_0 x + b_m \sin m\omega_0 x\right) \qquad (5)$$

Instead of using the standard integral formulas for the coefficients, a_m and b_m, it is easier to proceed using certain known expansions. From the binomial formula[2]

$$(1+z)^p = \sum_{m=0}^{\infty}\binom{p}{m} z^m \qquad (6)$$

valid for $|z|<1$ and p any real number,* we obtain

$$t = \sum_{m=0}^{\infty}\binom{1/2}{m} M^m \sin^m \omega_0 x \qquad (7)$$

There are two expansions[3] for $\sin^m \omega_0 x$, one for m even, the other for m odd. These are

$$\sin^{2k}\theta = \frac{(-1)^k}{2^{2k-1}}\left\{\sum_{j=0}^{k-1}(-1)^j\binom{2k}{j}\cos 2(k-j)\theta + (-1)^k\frac{1}{2}\binom{2k}{k}\right\} \qquad (8)$$

and

$$\sin^{2k+1}\theta = \frac{(-1)^k}{2^{2k}}\sum_{j=0}^{k}(-1)^j\binom{2k+1}{j}\sin\left(2[k-j]+1\right)\theta \qquad (9)$$

Substituting (8) and (9) in (7) and comparing with (5), we find that $a_1 = 0$ and

$$a_0 = \sum_{k=0}^{\infty}\alpha_{2k}M^{2k} \quad ; \quad \alpha_{2k} = \frac{1}{2^{2k}}\binom{1/2}{2k}\binom{2k}{k}$$
$$b_1 = M\sum_{k=0}^{\infty}\beta_{2k}M^{2k} \quad ; \quad \beta_{2k} = \frac{1}{2^{2k}}\binom{1/2}{2k+1}\binom{2k+1}{k} \qquad (10)$$

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* $\binom{p}{m} = p(p-1)(p-2)\cdots(p-m+1)$

Consequently, we can write for λ (as defined in 4) an exact expression, viz.

$$\lambda = \frac{b_1^2}{a_0^2} = M^2 R^2 \tag{11}$$

in which

$$R = \frac{\sum_{k=0}^{\infty} \beta_{2k} M^{2k}}{\sum_{k=0}^{\infty} \alpha_{2k} M^{2k}} = \sum_{k=0}^{\infty} \gamma_{2k} M^{2k} \tag{12}$$

The coefficients γ_{2k} are determined from the reoccurrence relationships

$$\beta_{2k} = \sum_{r=0}^{k} \gamma_{2r} \alpha_{2k-2r} \quad ; \quad k = 0, 1, 2, \ldots \tag{13}$$

Solving (13) for γ_0, γ_2, γ_4, and γ_6, we find that (11) becomes

$$\lambda = \frac{M^2}{4} \left\{ 1 + \frac{5}{32} M^2 + \frac{15}{256} M^4 + \frac{1,965}{65,536} M^6 + \ldots \right\}^2 \tag{14}$$

Comparing (14) with (4), we see that the first correction term $5/32\, M^2$ is about 3% of 1 for 50% modulation and would change λ by 6%.

REMARKS

Additional studies of the harmonic distortion introduced by non-linear operations could be based on the techniques referred to in the bibliography of Reference 4.

REFERENCES

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1. [redacted] Memorandum Evaluation of Microdensitometer Resolution Characteristics by Sine-Wave and Edge Trace Techniques

2. K. Knopp The Theory and Application of Infinite Series Blackie & Son Limited London 1948 p. 209

3. L. Jolley Summation of Series Dover p. 123

4. J.C. Hsu Integral Representation of Zero-Memo Nonlinear Functions Bell System Technical Journal 41 (6) 1813 (1962)